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ENERGY CORRECTIONS AND PERSISTENT PERTURBATION EFFECTS IN CONTINUOUS SPECTRA

II. THE PERTURBED STATIONARY STATES

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Synopsis

The paper continues the systematic investigation undertaken earlier [1]) of a class of perturbations of continuous energy spectra producing, in addition to scattering and reaction processes, self-energy and cloud effects which affect at all times the motion of wave packets and which are of the type occurring in the quantum theory of interacting fields. The main result presented here is the explicit determination of the perturbed stationary states. The formula obtained is used to express the motion of wave packets and to connect it with its asymptotic properties for large times as established earlier. A special example is treated as illustration of the general method and a crude preliminary discussion is given of the aspect under which the renormalization program will present itself in the present formalism

1. *Introduction.* In a previous paper [1]), to be referred to hereafter as I, we have introduced and analyzed a class of quantum-mechanical perturbations of continuous energy spectra characterized by the most important formal properties through which the interaction energies encountered in the quantum theory of fields differ from the interactions studied in conventional collision theory (theory of scattering and reaction processes, S-matrix theory). We have established in I that the perturbations there considered give rise, in line with the physical situation expected to occur for interacting fields and in contrast with the case of ordinary collisions, not only to scattering and reaction processes but also to permanent effects modifying the motion of arbitrary wave packets at all times. These effects consist of energy corrections in the continuous spectrum (self-energy effects in the terminology of field theory) and of what may be called cloud effects, i.e. the persistent admixture to each unperturbed stationary state of a "cloud" of other unperturbed stationary states, even in the asymptotic motion of wave packets long before or after all scattering and reaction processes have taken place. These self-energy and cloud effects have been derived to general order in the perturbation by a method more suitable for this purpose than the usual methods of field theory. They are contained in the perturbed energy $E(\alpha)$ and the "asymptotically stationary" state $|\alpha\rangle_{as}$ associated in I to each unperturbed

stationary state $|\alpha\rangle$. Their defining equations are (I.4.1.)*) for $E(\alpha)$, $-E=E(\alpha)$ is the root of this equation —, and (I.5.12) for the state $|\alpha\rangle_{as}$. A further result derived in I is the S-matrix formulae (I.6.4), (I.6.5.) expressing the connection between the asymptotic motions of wave packets before and after all transient (i.e. scattering and reaction) processes have taken place.

The main object of the present paper is to derive for the perturbations considered in I an explicit expression to general order for the perturbed stationary states. This is achieved in the next section. As an application of this result we then consider the exact expression for the time evolution of a wave packet under the perturbation and derive from it the asymptotic motion for large times, thus obtaining in another way the results established in I without knowledge of the explicit solution of the Schrödinger equation. Such is the contents of Section 3. Two more or less complementary examples of the general formalism are then considered. The first one (Section 4), only mentioned very briefly, is the case of perturbations producing transient effects only, no self-energy or cloud effects; the general equations then reduce to the well known results of collision theory. The second example (Section 5), dealt with in more detail, is a type of perturbation producing non-vanishing self-energy and cloud effects, but sufficiently simple to be calculated in closed form, and thus particularly well suited as illustration of the developments forming the main contribution of I and of the present paper. Finally, in the last section, some indications without aim at completeness are given on the aspect taken in the present formalism by the well known renormalization program of quantum field theory, and a few comparative remarks are made on a very recent paper by De Witt [2]) devoted to a subject closely related to ours.

The definitions, notations and results of I will be used throughout. In this paper as in I we adopt a notation adapted to the assumption that all quantum numbers defining the unperturbed states $|\alpha\rangle$ are continuous. This excludes the consideration of polarization indices, spin indices, etc. The extension of the formalism to include such discrete quantum numbers would present no essential difficulty. The main difference with the case considered here and in I would be that the diagonal part of a product $VA_1V \ldots A_nV$ (V is the perturbation and $A_1, \ldots A_n$ are operators diagonal in the $|\alpha\rangle$-representation) should be defined by (I.2.4) with the δ-function referring to the continuous quantum numbers alone, F_1 being consequently a finite matrix in the polarization indices. Similarly the quantities $\mathcal{G}_l(\alpha)$ and $D_l(\alpha)$ are finite matrices which can be diagonalized for each l, a step necessary for the definition of the perturbed energies $E(\alpha)$ and of the asymptotically stationary states $|\alpha\rangle_{as}$ as well as for the definition of the true stationary states by the method of the next section. Except for this short remark we

*) By Equation (I.4.1) we mean Equation (4.1) of paper I.

leave out of consideration as we did in I the complications connected with polarization indices. Another point which is assumed here as it was in I is the convergence of all expansions in powers of the perturbation which are used in the course of our derivations. This assumption excludes for example the possibility of bound states.

2. *The perturbed stationary states.* The resolvent operator R_l extensively used in I is also a convenient tool for determining the stationary states of the perturbed hamiltonian $H + \lambda V$. From its definition (I.3.1) one obtains for the spectral resolution of $H + \lambda V$ the equation

$$H + \lambda V = \int_{-\infty}^{+\infty} E\, P_E\, \mathrm{d}E,$$

where

$$P_E = (2\pi i)^{-1} \lim_{\eta\to 0} (R_{E+i\eta} - R_{E-i\eta}),\ \eta > 0. \tag{2.1}$$

The identity (I.3.17) gives on the other hand

$$R_{E+i\eta} - R_{E-i\eta} = 2\eta i\, R_{E\pm i\eta}\, R_{E\mp i\eta} \tag{2.2}$$

Here and in subsequent equations upper (lower) signs must be taken together. The limiting value of the right hand side of (2.2) for $\eta \to 0$ is determined by means of the following identities, which are quite easy to establish. If the matrix element $\langle\alpha|B|\alpha'\rangle$ considered as function of α and α' has no $\delta(\alpha - \alpha')$ — singularity, one has

$$\lim_{\eta\to 0} \eta\, D_{E\pm i\eta}\, B\, D_{E\mp i\eta} = 0. \tag{2.3}$$

If on the contrary B is diagonal in the $|\alpha\rangle$-representation this equation is replaced by *)

$$\lim_{\eta\to 0} \eta D_{E\pm i\eta}\, B D_{E\mp i\eta} = \pi\, NB\, \delta(H - E - \lambda^2 K_E),\ \eta > 0, \tag{2.4}$$

where N is the diagonal operator defined in terms of (I.5.8) by

$$N\,|\alpha\rangle = N(\alpha)\,|\alpha\rangle.$$

By separating in (2.2) the diagonal parts occurring in the product of two resolvents and by taking (2.3) and (2.4) into account one obtains

$$\lim \eta\, R_{E\pm i\eta}\, R_{E\mp i\eta} = \pi \lim R_{E\pm i\eta}\, D^{-1}_{E\pm i\eta}\, \delta(H - E - \lambda^2 K_E)\, D^{-1}_{E\mp i\eta}\, R_{E\mp i\eta}, \tag{2.5}$$

or, with the familiar notation $A_{E\pm i0} = \lim_{\eta\to 0} A_{E\pm i\eta}$ for $\eta > 0$,

$$\begin{aligned} P_E = {} & [1 + \Sigma_{n=1}^{\infty} \{(-\lambda\, D_{E\pm i0} V)^n\}_{nd}]. \\ & \delta(H - E - \lambda^2 K_E).[1 + \Sigma_{n'=1}^{\infty} \{(-\lambda V D_{E\mp i0})^{n'}\}_{nd}]. \end{aligned} \tag{2.6}$$

*) The last equation of I and equation (I.6.13) contain misprints whose correction is found hereunder in (2.4) and (2.5).

We now define

$$|\alpha\rangle_\pm = [N(\alpha)]^{\frac{1}{2}}.[1 + \Sigma_{n=1}^{\infty} \{(-\lambda D_{E(\alpha)\pm i0}\, V)^n\}_{nd}]\, |\alpha\rangle \tag{2.7}$$

and denote by ${}_\pm\langle\alpha|$ the hermitian conjugate state vector of $|\alpha\rangle_\pm$.

An easy transformation of (2.6) gives

$$P_E = \int |\alpha\rangle_\pm\, \delta[E - E(\alpha)]\, d\alpha\, {}_\pm\langle\alpha| \tag{2.8}$$

and consequently for the spectral resolution

$$H + \lambda V = \int |\alpha\rangle_\pm\, E(\alpha)\, d\alpha\, {}_\pm\langle\alpha| \tag{2.9}$$

As will now be established, these relations readily imply that the states $|\alpha\rangle_+$ (the states $|\alpha\rangle_-$) form a complete orthonormal set of eigenstates of the hamiltonian $H + \lambda V$; in formulae

$$(H + \lambda V)\, |\alpha\rangle_\pm = E(\alpha)\, |\alpha\rangle_\pm, \quad {}_\pm\langle\alpha \mid \alpha'\rangle_\pm = \delta(\alpha - \alpha') \tag{2.10}$$

where as always upper (lower) signs have to be taken together. The proof runs as follows. A well known property of spectral resolutions is expressed by the identity

$$P_E . P_{E'} = \delta(E - E')\, P_E.$$

Multiplying this equation on the left by the operator

$$N^{-\frac{1}{2}}\, [1 + \Sigma_1^\infty \{(-\lambda D_{E\pm i0} V)^n\}_{nd}]^{-1}$$

and on the right by

$$[1 + \Sigma_1^\infty \{(-\lambda V D_{E'\mp i0})^{n'}\}_{nd}]^{-1}\, N^{-\frac{1}{2}}$$

one obtains by means of (2.8)

$$\int |\alpha\rangle\, \delta[E - E(\alpha)]\, \mathrm{d}\alpha\, {}_\pm\langle\alpha \mid \alpha'\rangle_\pm\, \mathrm{d}\alpha'\, \delta[E' - E(\alpha')]\, \langle\alpha'| =$$
$$= \delta(E - E') \int |\alpha\rangle\, \delta[E - E(\alpha)]\, \mathrm{d}\alpha\, \langle\alpha|.$$

This is further reduced through multiplication on the left by $\langle\alpha_1|$ and on the right by $|\alpha_1'\rangle$:

$$\delta[E - E(\alpha_1)]\;\; \delta[E' - E(\alpha_1')]\;\; {}_\pm\langle\alpha_1\, |\alpha_1'\rangle_\pm = \delta(E - E')\;\; \delta[E - E(\alpha_1)]\; \delta(\alpha_1 - \alpha_1').$$

Integration over all values of E and E' for fixed α_1 and α_1' and application of (2.9) gives then (2.10).

3. *The asymptotic motion of wave packets.* The two complete sets of eigenstates obtained in the foregoing section provide us with two representations

$$\varphi(t) = \int |\alpha\rangle_\pm\, \exp\, [-\, it\, E(\alpha)]\; c_\pm(\alpha)\, \mathrm{d}\alpha \tag{3.1}$$

for an arbitrary solution of the time-dependent Schrödinger equation

$$i\, \partial\varphi/\partial t = (H + \lambda V)\varphi.$$

We have put $\hbar = 1$. The present section gives the connection between these representations and the results of I (Sections 5 and 6) on the asymptotic motion of wave packets. The following equations will be established

$$\left.\begin{array}{l}\lim_{t\to\mp\infty}[\varphi(t)-\int|\alpha\rangle_{as}\exp[-it\,E(\alpha)]\,c_{\pm}(\alpha)\,d\alpha]=0\\ \lim_{t\to+\infty}[\varphi(t)-\int|\alpha\rangle_{as}\exp[-it\,E(\alpha)]\,d\alpha\,\langle\alpha|S|\alpha'\rangle\,c_{+}(\alpha')\,d\alpha']=0\\ \lim_{t\to-\infty}[\varphi(t)-\int|\alpha\rangle_{as}\exp[-it\,E(\alpha)]\,d\alpha\,\langle\alpha|S^{*}|\alpha'\rangle\,c_{-}(\alpha')\,d\alpha']=0\end{array}\right\}\quad(3.2)$$

The limits are taken with the meaning that a state vector approaches zero when its norm (length) does. We note from (3.2) that $|\alpha\rangle_{+}$ corresponds to the situation where all scattering and reaction events produced by V contribute only to the outgoing wave components of the stationary state, while in $|\alpha\rangle_{-}$ they contribute to the incoming waves alone.

If two vectors $\varphi(t)$, $\psi(t)$ have norms approaching one for $t\to\pm\infty$, the equation

$$\lim_{t\to\pm\infty}[\varphi(t)-\psi(t)]=0$$

is equivalent to

$$\lim_{t\to\pm\infty}\langle\psi(t)\mid\varphi(t)\rangle=1.$$

To establish (3.2) we are thus led to calculate for $t\to\pm\infty$ the limiting value of a scalar product $\langle\psi(t)\mid\varphi(t)\rangle$ with $\varphi(t)$ of the form (3.1) and $\psi(t)$ of the form

$$\psi(t)=\int|\alpha\rangle_{as}\exp[-it\,E(\alpha)]\,\gamma(\alpha)\,d\alpha \qquad (3.3)$$

We carry out this calculation considering the upper sign in (3.1). The case of the lower sign is completely analogous. According to the respective definitions (I.5.12) and (2.7) of $|\alpha\rangle_{as}$ and $|\alpha\rangle_{+}$ we have

$$\begin{aligned}\langle\psi(t)\mid\varphi(t)\rangle&=\int[N(\alpha)\,N(\alpha')]^{\frac{1}{2}}\exp[it\,(E(\alpha)-E(\alpha'))]\\ \langle\alpha|\,[1&+\Sigma_{n=1}^{\infty}\{(-\lambda VY_{a}\,D_{E(a)\pm i0})^{n}\}_{nd}]\,\cdot\\ &\cdot[1+\Sigma_{n'=1}^{\infty}\{(-\lambda D_{E(\alpha')+i0}\,V)^{n'}\}_{nd}]\,|\alpha'\rangle\,\gamma^{*}(\alpha)\,c_{+}(\alpha')\,d\alpha\,d\alpha'.\end{aligned}\qquad(3.4)$$

In the product $Y_{a}\,D_{E(a)\pm i0}$ the choice of sign is irrelevant (I, Section 4). The limiting value of (3.4) for $t\to\pm\infty$ originates from the terms in the integrand which are singular at $E(\alpha)=E(\alpha')$. They are of two types. Firstly the diagonal part of the whole operator comprised in the matrix element gives

$$\langle\alpha|\,\{\ldots\}_{d}\,|\alpha'\rangle=[N(\alpha)]^{-1}\,\delta(\alpha-\alpha') \qquad (3.5)$$

as was established in I, Equations (I.5.18) and following. Its contribution to (3.4) is time-independent. Secondly there are the further diagonal parts to be separated in the matrix element. Since $Y_{a}\,D_{E(a)\pm i0}$ is non singular, the only singularities which can occur originate from terms where

$$[1+\Sigma_{n=1}^{\infty}\{(-\lambda VY_{a}\,D_{E(a)\pm i0})^{n}\}_{nd}]$$

appears as a whole within the diagonal part. One obtains by summing over all separations satisfying this condition the expression

$$\langle\alpha|\,\{[1 + \Sigma_{n=1}^{\infty}\{(-\lambda V Y_a D_{E(\alpha)\pm i0})^n\}_{nd}]\cdot$$
$$[1 + \Sigma_{n'=1}^{\infty}\{(-\lambda D_{E(\alpha')+i0} V)^{n'}\}_{nd}]\}_d\, D_{E(\alpha')+i0}$$
$$[-\lambda V + \lambda^2\{V R_{E(\alpha')+i0} V\}_{nd}]\,|\alpha'\rangle =$$
$$= -\lambda\,[N(\alpha)]^{-1} D_{E(\alpha')+i0}(\alpha)\,\langle\alpha|\,V - \lambda\{V R_{E(\alpha')+i0} V\}_{nd}\,|\alpha'\rangle. \tag{3.6}$$

The singular factor is

$$D_{E(\alpha')+i0}(\alpha) = [\varepsilon(\alpha) - E(\alpha') - i0 - \lambda^2 \mathcal{G}_{E(\alpha')+i0}(\alpha)]^{-1} =$$
$$= [E(\alpha) - E(\alpha') - i0]^{-1} N(\alpha) + \text{regular terms}, \tag{3.7}$$

where $N(\alpha)$ occurs through (I.5.8.). Its contribution for large times follows from

$$\lim \frac{\exp[it\,(E(\alpha) - E(\alpha'))]}{E(\alpha) - E(\alpha') - i0} = \begin{cases} 2\pi i\,\delta[E(\alpha) - E(\alpha')] \text{ for } t \to +\infty, \\ 0 \text{ for } t \to -\infty. \end{cases} \tag{3.8}$$

Gathering the results (3.5) to (3.8) one finds

$$\lim_{t\to-\infty} \langle\psi(t)\mid\varphi(t)\rangle = \int \gamma^*(\alpha)\, c_+(\alpha)\, d\alpha, \tag{3.9}$$

and, in view of the expression (I.6.5) of the S-matrix,

$$\lim_{t\to+\infty} \langle\psi(t)\mid\varphi(t)\rangle = \int \gamma^*(\alpha)\, c_+(\alpha)\, d\alpha - 2\pi i\,\lambda \int [N(\alpha)\, N(\alpha')]^{\frac{1}{2}}$$
$$\delta\,[E(\alpha) - E(\alpha')]\,\langle\alpha|V - \lambda\{V R_{E(\alpha)+i0} V\}_{nd}\,|\alpha'\rangle\,\gamma^*(\alpha)\, c_+(\alpha')\, d\alpha\, d\alpha' =$$
$$= \int \gamma^*(\alpha)\, d\alpha\,\langle\alpha|S|\alpha'\rangle\, d\alpha'\, c_+(\alpha'). \tag{3.10}$$

One would find similarly by taking the lower sign in (3.1)

$$\lim_{t\to-\infty} \langle\psi(t)\mid\varphi(t)\rangle = \int \gamma^*(\alpha)\, d\alpha\,\langle\alpha|S^*|\alpha'\rangle\, d\alpha'\, c_-(\alpha'), \tag{3.11}$$
$$\lim_{t\to+\infty} \langle\psi(t)\mid\varphi(t)\rangle = \int \gamma^*(\alpha)\, c_-(\alpha)\, d\alpha. \tag{3.12}$$

We may incidentally remark that comparison of (3.9) and (3.11) for arbitrary $\gamma(\alpha)$ gives

$$c_+(\alpha) = \int \langle\alpha|\, S^*\, |\alpha'\rangle\, d\alpha'\, c_-(\alpha') \tag{3.13}$$

whereas (3.10) and (3.12) provide the inverse relation

$$c_-(\alpha) = \int \langle\alpha|\, S\, |\alpha'\rangle\, d\alpha'\, c_+(\alpha'). \tag{3.14}$$

Since the function $c_+(\alpha)$ or the function $c_-(\alpha)$ may be arbitrarily chosen the two last equations imply the unitarity of the S-matrix, a property which was not explicitly established in I.

It is now an easy matter to complete the proof of Equations (3.2). One assumes $\varphi(t)$ normalized to one and one successively selects for $\psi(t)$ the four vectors of form (3.3) appearing in (3.2). One then verifies that the norm of

$\psi(t)$ always approaches one for $t \to \pm \infty$; this follows readily from the asymptotic orthonormality of the states $|\alpha\rangle_{as}$ (last equation of Section 5 in I) and from the unitary character of the S-matrix. One finally establishes by means of Equations (3.9) to (3.12) that $\langle\psi(t) \mid \varphi(t)\rangle$ has the limit one for infinite times.

We shall end this section with a remark on the asymptotically stationary states $|\alpha\rangle_{as}$. The definition of these states in I, Equation (I.5.12), may have appeared rather arbitrary and found only a justification a posteriori in the fact that they provide a complete and consistent description for the asymptotic motion of wave packets. Now however new light is thrown on this definition by its close analogy with the expression (2.7) of the true stationary states $|\alpha\rangle_{\pm}$. The difference lies in the occurrence of the projection operator Y_a in front of each factor V, i.e. in a restriction of the set of states $|\alpha'\rangle$ coupled to the unperturbed state $|\alpha\rangle$. The restriction is to those states which play an effective role in the eigenvalue of diagonal parts for $|\alpha\rangle$. They are also the states which contribute to the properties of wave packets

$$\varphi = \int |\alpha\rangle_{\pm}\, \mathrm{d}\alpha\, c(\alpha)$$

even for the most incoherent distribution of the phases of the amplitudes $c(\alpha)$, and they may therefore intuitively be pictured as the states $|\alpha'\rangle$ belonging to the "cloud" persistently attached to $|\alpha\rangle$ by virtue of the perturbation. It is an interesting and satisfactory result of the general theory that the restriction mentioned above results in the same asymptotically stationary states $|\alpha\rangle_{as}$ and thus in the same persistent cloud effects whether it is applied to the state $|\alpha\rangle_{+}$ with its outgoing nature of all scattered waves, or to the state $|\alpha\rangle_{-}$ with incoming scattered waves. We have here to do with a very significant fact which clarifies the physical meaning of the formal properties described in Section 4 of I.

4. *Perturbations without persistent effects.* It is obvious that the foregoing results apply in particular to the familiar case of perturbations which do not produce persistent effects because mathematically they do not give rise to any diagonal parts of the type defined in I, Section 2. These are the perturbations causing scattering and reaction processes only. For them one simply has to put in the results of I and of the foregoing sections

$$\mathcal{G}_l = 0,\ E(\alpha) = \varepsilon(\alpha),\ N(\alpha) = 1,\ |\alpha\rangle_{as} = |\alpha\rangle,$$

and to drop brackets of the type $\{\ldots\}_{nd}$. Although this case has been repeatedly studied in detail [3]) it may be worth noting that the method we used in the foregoing sections, when applied to it, gives an especially brief derivation of the perturbed stationary states and of the S-matrix. Our method is therefore of some interest also for ordinary collision theory.

5. *A simple type of perturbation with persistent effects.* The type of perturbation considered in the present section as illustration of the general formalism is amenable to an exact treatment in consequence of its very special mathematical structure. It has often been used on this ground as simplified model for various physical problems, in particular for resonance scattering and metastable states (theory of the line-width) [4]), and more recently by L e e [5]) for mass and coupling constant renormalization in quantum field theory *). We present this type of perturbation in a slightly more general form than usual. The generalization, which is mathematically trivial, amounts physically to the inclusion of recoil effects.

The unperturbed system of hamiltonian H is defined as having two continuous families of stationary states. The states of the first one, characterized by the presence of one particle (the V-particle of Lee), are labelled by its momentum $\mathbf{q}$ and are denoted by $|\mathbf{q}\rangle$. The states of the second family, denoted by $|\mathbf{q}, \mathbf{k}\rangle$, have two distinct particles present (the N- and θ-particles of Lee) of momenta $\mathbf{q}$ and $\mathbf{k}$. The unperturbed energy is defined by

$$H\,|\mathbf{q}\rangle = \varepsilon(\mathbf{q})\,|\mathbf{q}\rangle,\ H\,|\mathbf{q}, \mathbf{k}\rangle = \varepsilon(\mathbf{q}, \mathbf{k})\ |\mathbf{q}, \mathbf{k}\rangle. \tag{5.1}$$

One usually supposes $\varepsilon(\mathbf{q}, \mathbf{k})$ to be the sum of two one-particle energies, but this is irrelevant for the mathematical handling of the perturbation problem. The unperturbed stationary states $|\mathbf{q}\rangle$, $|\mathbf{q}, \mathbf{k}\rangle$, which correspond to the states $|\alpha\rangle$ of the general formalism, form two continuous families of dimensions 3 and 6 respectively. Orthonormality is expressed by

$$\left.\begin{aligned} &\langle\mathbf{q}\mid\mathbf{q}'\rangle = \delta(\mathbf{q}-\mathbf{q}'),\ \langle\mathbf{q}, \mathbf{k}\mid\mathbf{q}'\rangle = 0,\\ &\langle\mathbf{q}, \mathbf{k}\mid\mathbf{q}', \mathbf{k}'\rangle = \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{k}-\mathbf{k}'). \end{aligned}\right\} \tag{5.2}$$

The perturbation V is defined by its matrix elements in the unperturbed representation:

$$\left.\begin{aligned} &\langle\mathbf{q}'\,|V|\,\mathbf{q}, \mathbf{k}\rangle = \langle\mathbf{q}, \mathbf{k}\,|V|\,\mathbf{q}'\rangle^* = v(\mathbf{q}, \mathbf{k})\ \delta(\mathbf{q}+\mathbf{k}-\mathbf{q}'),\\ &\langle\mathbf{q}\,|V|\,\mathbf{q}'\rangle = \langle\mathbf{q}, \mathbf{k}\,|V|\,\mathbf{q}', \mathbf{k}'\rangle = 0 \end{aligned}\right\} \tag{5.3}$$

It allows for emission of a θ-particle by a V-particle which at the same time transforms into a N-particle and for the inverse process, with conservation of momentum.

We now show that the results of I and of the present paper, when applied to this very special type of perturbation, give immediately the exact diagonalization of the hamiltonian $H + \lambda V$ and the exact expression of the S-matrix. One first notices that irreducible diagonal parts occur only for products VAV containing two factors V and that they have the value

$$\{VAV\}_{id}\,|\mathbf{q}'\rangle = |\mathbf{q}'\rangle \int A(\mathbf{q}, \mathbf{k})\,|v(\mathbf{q}, \mathbf{k})|^2\,\delta(\mathbf{q}+\mathbf{k}-\mathbf{q}')\,d\mathbf{q}\,d\mathbf{k},$$
$$\{VAV\}_{id}\,|\mathbf{q}, \mathbf{k}\rangle = 0.$$

$A(\mathbf{q}, \mathbf{k})$ is the eigenvalue for the state $|\mathbf{q}, \mathbf{k}\rangle$ of the operator A assumed to be

*) It is not clear whether Lee realized the very close resemblance of his field-theoretical model with the model introduced by Dirac for resonance scattering.

diagonal in the $|\mathbf{q}\rangle$, $|\mathbf{q}, \mathbf{k}\rangle$-representation. Consequently the operator $\mathcal{G}_l$ implicitly given by (I.3.13) can be calculated exactly and is found to have the following eigenvalues for $|\mathbf{q}'\rangle$ and $|\mathbf{q}, \mathbf{k}\rangle$:

$$\left.\begin{aligned} \mathcal{G}_l(\mathbf{q}') &= \int [\varepsilon(\mathbf{q}, \mathbf{k}) - l]^{-1} . |v(\mathbf{q}, \mathbf{k})|^2 . \delta(\mathbf{q} + \mathbf{k} - \mathbf{q}')\, d\mathbf{q}\, d\mathbf{k}, \\ \mathcal{G}_l(\mathbf{q}, \mathbf{k}) &= 0. \end{aligned}\right\} \tag{5.4}$$

The expression of the operators K_E and J_E follows immediately by application of the definition (I. 3.20). The perturbed energy values $E(\alpha)$ of the general theory are here $E(\mathbf{q}, \mathbf{k}) = \varepsilon(\mathbf{q}, \mathbf{k})$ and the root $E = E(\mathbf{q}')$ of the numerical equation

$$\varepsilon(\mathbf{q}') - E - \lambda^2 K_E(\mathbf{q}') = 0. \tag{5.5}$$

This root is supposed to be unique. The physically important condition $J_{E(\alpha)}(\alpha) = 0$, introduced in I, Section 4, to characterize the perturbations which give rise to self-energy and cloud effects as opposed to those producing dissipative effects, takes here the form

$$v(\mathbf{q}, \mathbf{k}) = 0 \text{ whenever } \mathbf{q} + \mathbf{k} = \mathbf{q}',\ \varepsilon(\mathbf{q}, \mathbf{k}) = E(\mathbf{q}'). \tag{5.6}$$

We assume it to be satisfied. It is instrumental in making our perturbation a meaningful model of field-theoretical interaction. When it is not satisfied one obtains a model for resonance scattering and metastable states [4]. The last important point to notice is that the non-diagonal part of any product $VA_1 V \ldots A_n V$ ($A_1, \ldots A_n$ diagonal in the $|\mathbf{q}\rangle$, $|\mathbf{q}, \mathbf{k}\rangle$-representation) with more than two factors V ($n \geqslant 2$) vanishes. It is for this reason that our general formulae give in the case at hand a closed expression for the perturbed stationary states and for the S-matrix. The stationary states are found from (2.7)

$$\left.\begin{aligned} &|\mathbf{q}'\rangle_+ = |\mathbf{q}'\rangle_- = [N(\mathbf{q}')]^{\frac{1}{2}} \left[|\mathbf{q}'\rangle - \lambda \int |\mathbf{q}, \mathbf{k}\rangle . \frac{v^*(\mathbf{q}, \mathbf{k})\, \delta(\mathbf{q}+\mathbf{k}-\mathbf{q}')}{\varepsilon(\mathbf{q}, \mathbf{k}) - E(\mathbf{q}')} d\mathbf{q}\, d\mathbf{k} \right], \\ &|\mathbf{q}, \mathbf{k}\rangle_\pm = |\mathbf{q}, \mathbf{k}\rangle - \lambda v(\mathbf{q}, \mathbf{k}) \,[\varepsilon(\mathbf{q} + \mathbf{k}) - \varepsilon(\mathbf{q}, \mathbf{k}) \mp i0 - \\ &\qquad\qquad - \lambda^2\, \mathcal{G}_{\varepsilon(\mathbf{q},\mathbf{k}) \pm i0}\, (\mathbf{q} + \mathbf{k})]^{-1} \\ &\left[|\mathbf{q} + \mathbf{k}\rangle - \lambda \int |\mathbf{q}', \mathbf{k}'\rangle \frac{v^*(\mathbf{q}', \mathbf{k}')\, \delta(\mathbf{q}' + \mathbf{k}' - \mathbf{q} - \mathbf{k})}{\varepsilon(\mathbf{q}', \mathbf{k}') - \varepsilon(\mathbf{q}, \mathbf{k}) \mp i0} d\mathbf{q}'\, d\mathbf{k}' \right], \end{aligned}\right\} \tag{5.7}$$

with

$$[N(\mathbf{q}')]^{-1} = 1 + \lambda^2 \int [\varepsilon(\mathbf{q}, \mathbf{k}) - E(\mathbf{q}')]^{-2}\, |v(\mathbf{q}, \mathbf{k})|^2\, \delta(\mathbf{q} + \mathbf{k} - \mathbf{q}')\, d\mathbf{q}\, d\mathbf{k}. \tag{5.8}$$

The asymptotically stationary states (I.5.12) simply reduce to

$$|\mathbf{q}'\rangle_{as} = |\mathbf{q}'\rangle_+ = |\mathbf{q}'\rangle_-,\ |\mathbf{q}, \mathbf{k}\rangle_{as} = |\mathbf{q}, \mathbf{k}\rangle. \tag{5.9}$$

Finally, for the S-matrix, the formula (I.6.5) gives

$$\left.\begin{aligned} &\langle \mathbf{q}\, |S|\, \mathbf{q}'\rangle = \delta(\mathbf{q} - \mathbf{q}'),\ \langle \mathbf{q}, \mathbf{k}\, |S|\, \mathbf{q}'\rangle = \langle \mathbf{q}'\, |S|\, \mathbf{q}, \mathbf{k}\rangle = 0, \\ &\langle \mathbf{q}, \mathbf{k}\, |S|\, \mathbf{q}', \mathbf{k}'\rangle = \delta(\mathbf{q} - \mathbf{q}')\, \delta(\mathbf{k} - \mathbf{k}') + 2\pi i\, \lambda^2\, \delta[\varepsilon(\mathbf{q}, \mathbf{k}) - \varepsilon(\mathbf{q}', \mathbf{k}')] \\ &\delta(\mathbf{q} + \mathbf{k} - \mathbf{q}' - \mathbf{k}')\ v^*(\mathbf{q}, \mathbf{k})\, v(\mathbf{q}', \mathbf{k}')\, [\varepsilon(\mathbf{q} + \mathbf{k}) - \varepsilon(\mathbf{q}, \mathbf{k}) - \\ &\qquad\qquad - \lambda^2\, \mathcal{G}_{\varepsilon(\mathbf{q},\mathbf{k}) + i0}(\mathbf{q} + \mathbf{k})]^{-1}. \end{aligned}\right\} \tag{5.10}$$

The perturbation problem is thus completely solved in terms of the root $E(\mathbf{q}')$ of the numerical equation (5.5). It is of course impossible to calculate this root explicitly for arbitrary forms of the functions $\varepsilon(\mathbf{q}')$, $\varepsilon(\mathbf{q}, \mathbf{k})$ and $v(\mathbf{q}, \mathbf{k})$.

6. *Preliminary remarks on the renormalization problem.* Since our general formalism will find its most important applications in quantum field theory one has to inquire as to the aspect it will give to the renormalization program which played such a central role in the successes of quantum electrodynamics[6]). No systematic study of this problem will be attempted here. We shall content ourselves with a few remarks, stressing one side of the problem on which our approach may eventually turn out to be more promising than the conventional ones.

The physical basis of the renormalization method in field theory lies in the fact that the interaction V inescapably affects all observations one can make on the field particles, so that we have no experimental access to what the theory calls the unperturbed system (hamiltonian H). Still, all the theory has to build upon is an hamiltonian $H + \lambda V$ composed of the term H belonging to this unobservable unperturbed system and a perturbation term V acting upon it. This remarkable situation *) has been met with by the remark that in view of the permanent presence of the interaction the constants, mass and charge, occurring in the two terms H and λV of the total hamiltonian are not necessarily equal to the corresponding measured quantities, so that a redefinition of them is required if one wants to use their measured values in comparing any theoretical prediction with experiment. Exploitation of this idea of mass and charge renormalization turned out to be possible and, beyond providing a method to circumvent mathematical divergence difficulties, it gave in quantum electrodynamics a brilliant explanation of new experimental facts. The formalism, either in the presentation of Dyson[7]) or in that of Källén[8]), and as far as electrodynamics is concerned, makes essential use of three so-called renormalization constants, one for the mass, one for the charge and a third one of more purely formal significance, the wave function renormalization constant. They are respectively called δm, Z_3 and Z_2 by Dyson, K, L and N by Källén. Dyson introduced an additional constant Z_1, but conjectured its identity with Z_2. The correctness of this conjecture was established by Ward[9]). All three constants δm, Z_3, Z_2 closely correspond to simple elements of the formalism developped in I and in the present paper. Z_3 and Z_2 are simply special cases of the coefficient $N(\alpha)$ introduced by (I.5.8); they are obtained by taking for $|\alpha\rangle$ a one-photon state or a one-electron state respectively. The mass renormalization δm appears in our non-covariant formalism as an energy renormalization when the unperturbed energy $\varepsilon(\alpha)$ is eliminated in terms of the perturbed energy $E(\alpha)$.

*) It is an interesting and unsolved problem to understand for which physical reasons the historical development has been such as to confront us with this situation.

One thus concludes (in contradiction with the too pessimistic statement at the end of I) that the renormalization program in its conventional form could be carried out on the basis of our equations as well as in the more conventional presentations.

There is however one aspect of the conventional renormalization scheme which is unsatisfactory on physical grounds and for the improvement of which our approach may offer new possibilities. While the conventional method duly avoids use of the unrenormalized values of mass and charge, it is not able to avoid completely the use of the unperturbed states $|\alpha\rangle$, although they are just as inaccessible to observation as the unrenormalized mass and charge values. It is true that in view of the so-called wave function renormalization $|\alpha\rangle$ enters the conventional formalism only through the combination $[N(\alpha)]^{\frac{1}{2}}\,|\alpha\rangle$, but despite its mathematical usefulness this combination has the drawback of corresponding to the same unobservable physical state as $|\alpha\rangle$ itself. Physically such states are "bare" particle states, whereas a consistent application of the renormalization idea would exclusively allow consideration of "dressed" particles, i.e. of particles surrounded by the clouds which the interaction permanently maintains around each of them. We now remark that it has been one of the main aims of I to give an explicit description of such cloud effects; this description is contained in the equation for the asymptotic stationary states $|\alpha\rangle_{as}$, Equation (I.5.12). The states $|\alpha\rangle_{as}$ occur as monochromatic components in the asymptotic motion of wave packets and they are thus observable, just as the plane wave states of a Schrödinger particle are observable even when an external potential is present in a limited region of space and produces ordinary scattering.

The asymptotically stationary states $|\alpha\rangle_{as}$ are the states which must be chosen instead of $|\alpha\rangle$ or $[N(\alpha)]^{\frac{1}{2}}\,|\alpha\rangle$ as basic representation for a more consistent development of the renormalization program, and the results of I and of the present paper, by the explicit attention they pay to the asymptotically stationary states, are likely to provide a more convenient starting point for this development than the conventional formulations of field theory. Here we can only mention this new standpoint of which we hope to work out the consequences elsewhere. We may however already remark that under this point of view the divergence difficulties will appear at least partly in a rather different form than in the formalisms of Dyson and Källén. The choice of $|\alpha\rangle_{as}$ as basic representation is very likely to remove some of the divergencies occurring in the conventional formulation, because it avoids consideration of Z_2. The divergencies here concerned are those originating from the fact that in a theory without cut-off the expansion of $|\alpha\rangle_{as}$ in the unperturbed states $|\alpha'\rangle$ cannot be expected to exist. The general situation in this respect will probably turn out to be analogous to what was established previously [10]) for a neutral scalar field interacting with a static point source. The states $|\alpha\rangle_{as}$ will probably span a separable Hilbert space S different from the separable Hilbert space S_0 spanned by the unperturbed states $|\alpha\rangle$, and one must expect the true stationary states $|\alpha\rangle_\pm$ to be contained in S rather than in S_0. For the understanding of these various remarks it may be good to have in mind that in field theory, for one-particle states, the $|\alpha\rangle_{as}$ must be identical with the true stationary states $|\alpha\rangle_\pm$ (themselves independent of the double sign). For many-particle states on the contrary $|\alpha\rangle_{as}$ should be essentially a product wave function of one-particle state vectors while the true stationary states $|\alpha\rangle_\pm$ will be much more involved since they contain all scattering and reaction effects.

Before closing these general and quite tentative considerations we have to mention a very recent paper by De Witt [2]) with an aim similar to ours, namely to extend conventional scattering theory to perturbations producing self-energy effects. Whereas the previous work in this direction, due to Pirenne and taken over by Gell-Mann and Goldberger [11]), payed attention to the energy shifts only, the paper of De Witt goes much further and introduces in addition a coefficient of "state vector renormalization", which is essentially our coefficient $N(\alpha)$. There is however no explicit discussion of the cloud effects modifying the motion of wave packets even for asymptotic times and manifesting themselves in our formalism through the fact that $|\alpha\rangle_{as}$ is not identical with $[N(\alpha)]^{\frac{1}{2}}\, |\alpha\rangle$, nor is mention made of the fact that the perturbation must satisfy special conditions (see I, in particular Section 4) in order for the conclusions of the paper to be valid. Still these conditions are very important, because for example they make all the difference between the behaviour of electrons in interaction with the photon field and the completely different (dissipative) behaviour of conduction electrons of a metal in interaction with the phonon field (field of elastic vibrations).

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